Section 4-5 Mathematics 108

**Exponential and Log Equations**

Solving problems with exponential and log expressions can be a bit like solving a puzzle. Sometimes a very difficult looking expression can be easily simplified.

By looking at a few examples, the main strategies can be illuminated.

There are two facts about these functions that can come in handy when trying to solve an equation.

1) Exponential and Log functions are one-to-one.

That means that when we have an expression

$$a^x = a^y$$

we know that as a consequence

$$x = y$$

Similarly if

$$\log_a x = \log_a y$$

then

$$x = y$$

So sometimes the challenge is to get the two expressions into one of these forms.

2) Because Exponential functions and Log functions are one-to-one, they have an inverse.

Example 1a)

$$5^x = 125$$

Strategy 1

$$5^x = 125 = 5^3 \rightarrow x = 3$$

Strategy 2

$$5^x = 125 \rightarrow \log_5 5^2 = \log_5 125$$
$$\log_5 5^x = x \log_5 5 = x \cdot 1 = x$$
$$\log_5 125 = \log_5 5^3 = 3 \log_5 5 = 3$$
$$or$$
$$\log_5 125 = \log_5 5 \cdot 5 \cdot 5 = \log_5 5 + \log_5 5 + \log_5 5 = 1 + 1 + 1 = 3$$

Either way

$$x = 3$$

Example 1b)

$$5^{2x} = 5^{x+1}$$
$$\log_5 5^{2x} = \log_5 5^{x+1}$$
$$2x \log_5 5 = (x+1) \log_5 5$$
$$2x = x + 1$$
$$x = 1$$

Example 2

$3^{x+2} = 7$

We could look for this:

$$3^{y} = 7$$
$$\log_{10} 3^{y} = \log_{10} 7$$
$$y \log_{10} 3 = \log_{10} 7$$
$$y = \frac{\log_{10} 7}{\log_{10} 3}$$
$$3^{x+2} = 3^{\frac{\log_{10} 7}{\log_{10} 3}}$$
$$x + 2 = \frac{\log_{10} 7}{\log_{10} 3}$$
$$x = \frac{\log_{10} 7}{\log_{10} 3} - 2$$

A more direct route might be

$$3^{x+2} = 7$$
$$\log_3 3^{x+2} = \log_3 7$$
$$(x+2)\log_3 3 = \log_3 7$$
$$x + 2 = \log_3 7$$
$$x = \log_3 7 - 2$$

Of course if we want to find an approximate solution with our calculator, we either use the first solutions, or modify the second with the change of base formula. Either way

$$x = \frac{\log_{10} 7}{\log_{10} 3} - 2 \approx -0.228756251$$

Example 3

Keep in mind the same strategy for solving other equations holds. Isolate the $x$ variable, by performing the same operation on each side of the equation. Then if necessary try to factor.

$$8e^{2x} = 20$$

$$\frac{8e^{2x}}{8} = \frac{20}{8}$$

$$e^{2x} = \frac{5}{2}$$

$$\ln e^{2x} = \ln\frac{5}{2}$$

$$2x = \ln\frac{5}{2}$$

$$x = \frac{\ln\frac{5}{2}}{2} \approx .4581453659$$

Example 4

$$e^{3-2x} = 4$$

$$\ln e^{3-2x} = \ln 4$$

$$3 - 2x = \ln 4$$

$$2x = 3 - \ln 4$$

$$x = \frac{3 - \ln 4}{2} \approx .8068528194$$

Example 5: A Quadratic Exponential equation

$$e^{2x} - e^{x} - 6 = 0$$
$$y = e^{x}$$
$$y^2 - y - 6 = 0$$
$$(y-3)(y+2) = 0$$
$$y = 3, -2$$
$$e^{x} = 3 \to x = \ln 3$$
$$e^{x} = -2 \to x = \ln -2$$

But the range of $e^{x}$ or the domain of ln is $x$>0, so -2 is not a solution.

Note the strategy here. We could just as easily solve this equation now:

$$2e^{4x} + 3e^{3x} - 4e^{2x} - 3e^{x} + 2 = 0$$

Does anyone recognize this from the homework 3.4 #81. $2x^4 + 3x^3 - 4x^2 - 3x + 2 = 0$

The solutions were $x = 1, -1, -2, \frac{1}{2}$

So the solutions to our problem are

$$e^{x} = 1$$
$$x = 0$$
$$e^{x} = \frac{1}{2}$$
$$x = \ln\frac{1}{2} = -\ln 2 \approx -.6931471806$$

Example 6

A question where factoring solves the problem

$$3xe^x + x^2e^x = 0$$
$$xe^x(3+x) = 0$$

So we find solutions where

$$x = 0$$
$$x+3=0 \rightarrow x=-3$$

$e^x = 0$ which has no solutions

Example 7, remember the laws of dealing with logs

$$\log a + \log b = \log ab$$
$$\log a - \log b = \log a/b$$
$$\log a^b = b\log a$$

$$\log\left(x^2+1\right) = \log(x-2) + \log(x+3)$$
$$\log\left(x^2+1\right) = \log\left[(x-2)(x+3)\right]$$
$$x^2+1 = (x-2)(x+3) = x^2+x-6$$
$$1 = x-6$$
$$x = 7$$

Example 8

Keep in mind that the log function is the inverse to an exponential function, so you can always transform this way:

a)

$$\ln x = 8$$
$$as$$
$$e^{\ln x} = e^8$$
$$x = e^8$$

b)

$$\log_2 (25 - x) = 3$$
$$2^{\log_2 (25-x)} = 2^3$$
$$25 - x = 8$$
$$x = 17$$

Example 9

Don't let required extra operations confuse you.

Breath deeply and do what you know will work

$$4 + 3\log_{10} (2x) = 16$$
$$3\log_{10} (2x) = 12$$
$$\log_{10} (2x) = 4$$
$$2x = 10^4$$
$$x = 5000$$

Example 10

Don't forget those laws of exponents

$$\log_{10}(x+2)+\log(x-1)=1$$
$$\log_{10}\left[(x+2)(x-1)\right]=1$$
$$\log_{10}\left(x^2+x-2\right)=1$$
$$x^2+x-2=10$$
$$x^2+x-12=0$$
$$(x+4)(x-3)=0$$
$$x=-4,3$$

But checking -4 we find

$$\log_{10}(-2)+\log(-5)=1$$

Does not work, so $x$=3 is the only solution.

Example 11

Sometimes only a numerical method, eg. graphing will work.

$$x^2=2\ln(x+2)$$

Try it first!

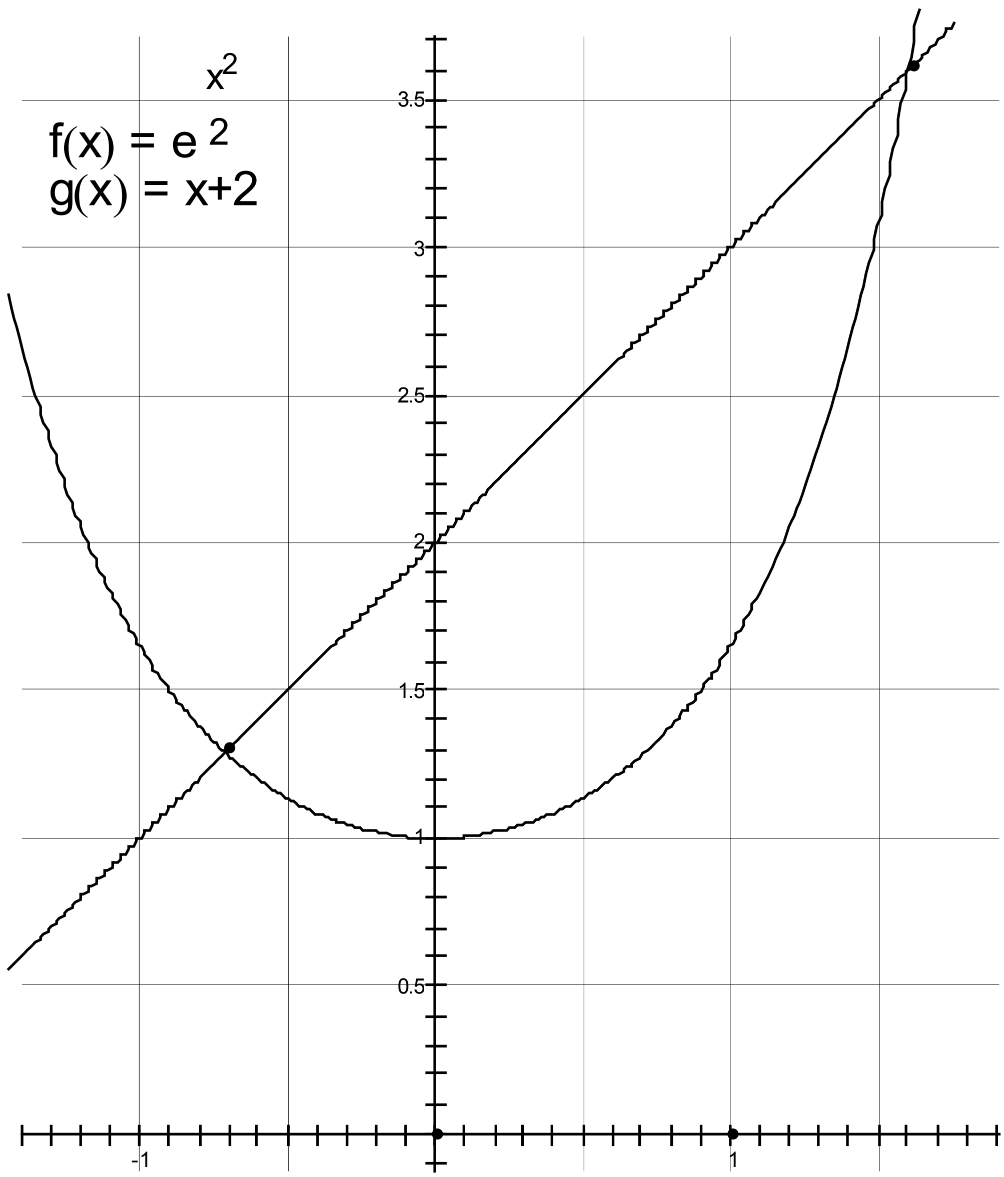
$$e^{x^2}=e^{2\ln(x+2)}=\left(e^{\ln(x+2)}\right)^2$$
$$e^{x^2/2}=x+2$$

This doesn't seem to be getting us anywhere.

However

| | $e^{x^2/2}$ | $x+2$ |
|---|---|---|
| 0 | 1 | 2 |
| 1 | $\sqrt{e}\approx 1.64$ | 3 |
| 2 | $e^2\approx 7$ | 4 |

As these functions go from 1 to 2 the first goes from 1.64 to about 7 but the second only goes from 3 to 4. That means they must intersect. Let's try a graph.




We can see here that there are two intersections.
One at about 1.65 and the other at about -.7

Using a Ti-83, graphing the function $f(x) = e^{x^2/2} - (x+2)$ and using the calculate zero's button we find these points are closer to.
-.7117437 and 1.6007061

Example 13

A sum of $5000 is invested at an interest rate of 5% per year,. Find the time required for the money to double if the interest is compounded (a) semi-annually and (b) continuously.

a)

$$P(y) = P_0\left(1+\frac{.025}{2}\right)^{2y}$$

$$10,000 = 5000\left(1+\frac{.025}{2}\right)^{2y}$$

$$2 = \left(1+\frac{.025}{2}\right)^{2y}$$

$$\log 2 = 2y\log\left(1+\frac{.025}{2}\right)$$

$$y = \log 2/2\log\left(1+\frac{.05}{2}\right) \approx 14.03$$

Which really means 14 years 6months

b)

$$P(y) = P_0 e^{.05y}$$

$$10,000 = 5000e^{.05y}$$

$$2 = e^{.05y}$$

$$\ln 2 = .05y$$

$$y = \ln 2/.05 \approx 13.8629$$

Which really means 13 years 315 days.